CS7800: Advanced Algorithms

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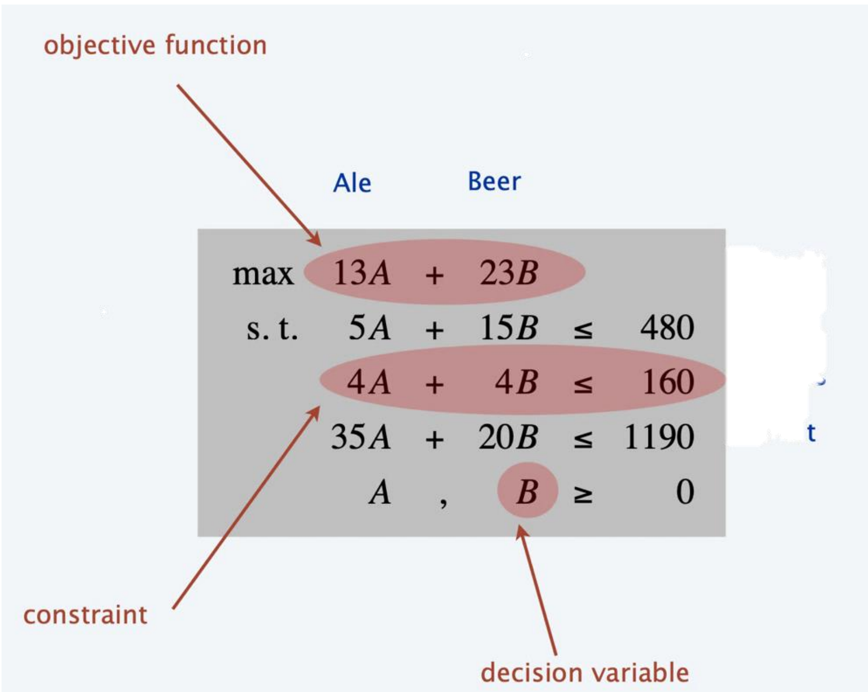
Linear Programming (Part 2)

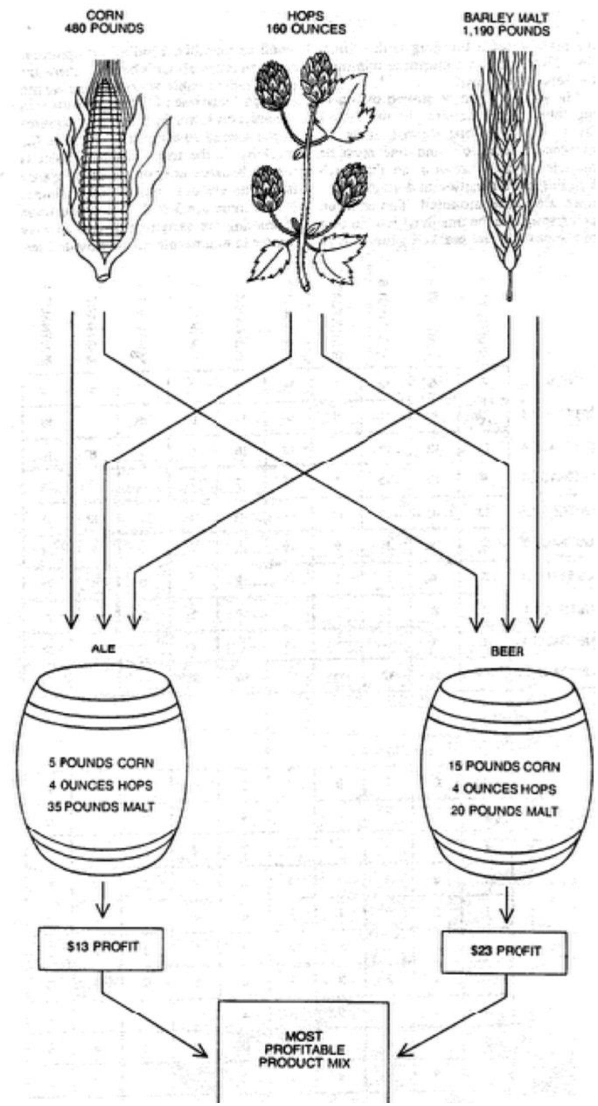
Duality

Slides credit: Kevin Wayne

Recall: The Brewery LP



Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit ($)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	



Duality

"Primal" problem:

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
& \text{s.t.} & 5A & + & 15B & \leq & 480 \\
& & 4A & + & 4B & \leq & 160 \\
& & 35A & + & 20B & \leq & 1190 \\
& & A & , & B & \geq & 0
\end{array}
$$

Goal 1: Find a lower bound on optimal value.

Easy: Any feasible solution provides one.

- $(A, B) = (34, 0) \Rightarrow z^* \geq 442$
- $(A, B) = (0, 32) \Rightarrow z^* \geq 736$
- $(A, B) = (7.5, 29.5) \Rightarrow z^* \geq 776$
- $(A, B) = (12, 28) \Rightarrow z^* \geq 800$

Duality

"Primal" problem:

(P)	max	$13A$	+	$23B$		
	s. t.	$5A$	+	$15B$	≤	480
		$4A$	+	$4B$	≤	160
		$35A$	+	$20B$	≤	1190
		A	,	B	≥	0

Goal 2: Find an upper bound on optimal value.

Ex 1: Multiply 2nd inequality by 6: $24A + 24B \leq 960$. This gives

$$z^* = 13A + 23B \leq 24A + 24B \leq 960.$$

Ex 2: Add two times 1st inequality to 2nd inequality:

$$z^* = 13A + 23B \leq 14A + 34B \leq 1120.$$

Duality

"Primal" problem:

(P)	max	$13A$	+	$23B$		
	s. t.	$5A$	+	$15B$	≤	480
		$4A$	+	$4B$	≤	160
		$35A$	+	$20B$	≤	1190
		A	,	B	≥	0

Goal 2: Find an upper bound on optimal value.

Ex 3: Add one time 1st inequality to two times 2nd inequality:

$$z^* = 13A + 23B \leq 13A + 23B \leq 800.$$

Recall lower bound: $(A, B) = (12, 28) \quad \Rightarrow z^* \geq 800$

Combining upper and lower bounds: $z^* = 800$.

Duality

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
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 & & A & , & B & \geq & 0
\end{array}
$$

- Let's try to generalize this idea...
- Idea: Add non-negative combination (C, H, M) of constraints s.t.

$$
\begin{aligned}
13A + 23B &\leq (5C + 4H + 35M)\ A \ + \ (15C + 4H + 20M)\ B \\
&\leq 480C + 160H + 1190M
\end{aligned}
$$

- But how do we find C, H, M?

Duality

$$
\begin{array}{llrcrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
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&\leq 480C + 160H + 1190M
\end{aligned}
$$

- But how do we find C, H, M? By writing another LP... 🤯

- Dual: Find the best upper bound with constraint coefficients.

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
& \text{s.t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
& & 15C & + & 4H & + & 20M & \geq & 23 \\
& & C & , & H & , & M & \geq & 0
\end{array}
$$

Duality: Economic Interpretation

- Brewer / primal: find optimal mix of beer and ale to maximize profit.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
& \text{s.t.} & 5A & + & 15B & \leq & 480 \\
& & 4A & + & 4B & \leq & 160 \\
& & 35A & + & 20B & \leq & 1190 \\
& & A & , & B & \geq & 0
\end{array}
$$

- Entrepreneur / dual: buy each resource from brewer at min cost.
  - C, H, M : unit price for corn, hops, malt.
  - Brewer won't sell if $5C + 4H + 3M < 13$.

$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
& \text{s.t.} & 5C & + & 4H & + & 35M & \geq & 13 \\
& & 15C & + & 4H & + & 20M & \geq & 23 \\
& & C & , & H & , & M & \geq & 0
\end{array}
$$

Duality: Basic Observations

- If the primal has n variables and m constraints, then the dual has m variables and n constraints.

$$
\begin{array}{llrcrcr}
\text{(P)} & \max & 13A & + & 23B & & \\
& \text{s.t.} & 5A & + & 15B & \leq & 480 \\
& & 4A & + & 4B & \leq & 160 \\
& & 35A & + & 20B & \leq & 1190 \\
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$$
\begin{array}{llrcrcrcr}
\text{(D)} & \min & 480C & + & 160H & + & 1190M & & \\
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& & C & , & H & , & M & \geq & 0
\end{array}
$$

Double dual

- Canonical form:

$$\text{(P)} \quad \max c^T x \quad \text{s.t.} \quad Ax \leq b, \quad x \geq 0$$

$$\text{(D)} \quad \min y^T b \quad \text{s.t.} \quad A^T y \geq c, \quad y \geq 0$$

- Property: The dual of the dual is the primal.
- Pf: Rewrite (D) as maximization in canonical form, take its dual.

$$\text{(D')} \quad \max \; -y^T b \quad \text{s.t.} \quad -A^T y \leq -c, \quad y \geq 0$$

$$\text{(DD)} \quad \min \; -c^T z \quad \text{s.t.} \quad -(A^T)^T z \geq -b, \quad z \geq 0$$

The Dual Recipe

Primal (P)	maximize
constraints	$a\,x = b_i$ $a\,x \le b$ $a\,x \ge b_i$
variables	$x_j \ge 0$ $x_j \le 0$ unrestricted

minimize	Dual (D)
y_i unrestricted $y_i \ge 0$ $y_i \le 0$	variables
$a^{T}y \ge c_j$ $a^{T}y \le c_j$ $a^{T}y = c_j$	constraints

- Pf: Rewrite in canonical form and take dual.

Strong Duality

- Theorem (Weak Duality): For any A, b, c if (P) and (D) are feasible, then max $\leq$ min.

$$
\begin{aligned}
\text{(P)} \quad & \max c^T x \\
& \text{s.t.} \quad Ax = b \\
& \qquad\quad x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\text{(D)} \quad & \min y^T b \\
& \text{s.t.} \quad A^T y \geq c
\end{aligned}
$$

- Theorem (Strong Duality): For any A, b, c if (P) and (D) are feasible, then max = min.

Examples: Single Source Shortest Path

- Input: Weighted directed graph G=(V, E), weight function $\ell: E \to R$ and vertex $s \in V$.
- Output: Length of the shortest path from s to every other vertex.

$$\begin{aligned} \text{maximize} \quad & \sum_{v} dist(v) \\ \text{subject to} \quad & dist(s) = 0 \\ & dist(v) - dist(u) \le \ell(u \to v) \quad \text{for every edge } u \to v \end{aligned}$$

- What's the dual?