Graph Coloring

Given an unweighted undirected graph $G=(V,E)$.

k vertex coloring: An assignment $\chi: V \to \{1,\dots,k\}$ such that for any $(u,v)\in E$, $\chi(u)\neq\chi(v)$.

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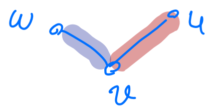
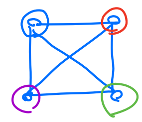
k edge coloring: An assignment $\chi: E\to\{1,\dots,k\}$ such that for any distinct $(u,v),(w,v)\in E$, $\chi(u,v)\neq\chi(w,v)$.



Thm: Every graph of max degree Δ admits a $\Delta+1$ vertex coloring, which can be found in $O(m)$ time.

pf: Go over vertices in an arbitrary order and color them. Since the # colors is more than degrees, there is one color available for every vertex that we process. $\square$

Remark: The complete graph on $\Delta+1$ vertices requires $\Delta+1$ colors.



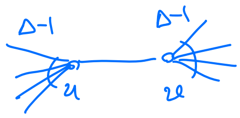
Next Q: Given k, G can we color G with k colors?

* $k=2$: Run BFS and assign 'red' to vertices of odd level, 'blue' to vertices of even level, and check if there's a conflict.

* $k\geq 3$: NP-hard

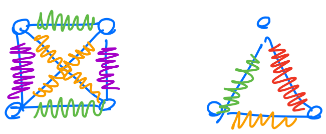
Thm: Every graph of max degree Δ admits a $2\Delta-1$ edge coloring, which can be found in $O(m\Delta)$ time.

pf. Go over the edges in arbitrary order. Upon visiting (u,v) find a feasible color, which must exist.



Obs: Any graph of max degree Δ needs at least Δ colors to be properly edge colored.

pf: All the edges of the vertex with max degree should be assigned different colors.



Thm (folklore): Any bipartite graph of max degree Δ can be edge colored using Δ colors.

Thm [Vizing '64]: Every graph of max degree Δ admits a $\Delta+1$ edge coloring.

* Given G, can we can edge color it using Δ colors? NP-hard [Holyer '81].

* Given G, how fast can we compute a $\Delta+1$ edge coloring?

[Vizing '64]: $O(mn)$ time

det.

[Gabow et al '85, Arjomandi '82] $\tilde{O}(m\sqrt{n})$

⋮

[Assadi, B, Bhattacharya, Costa, Solomon, Zhang Oct '24] $\tilde{O}(m)$

randomized

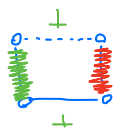
Thm (folklore): Any bipartite graph of max degree Δ can be edge colored using Δ colors in $O(mn)$ time.

Pf: Define a partial coloring to be an assignment $\chi : E \to \{1, \dots, \Delta, \perp\}$ such that for any two incident edges (u,v), (w,v) not colored $\perp$, we have $\chi(u,v) \neq \chi(w,v)$.

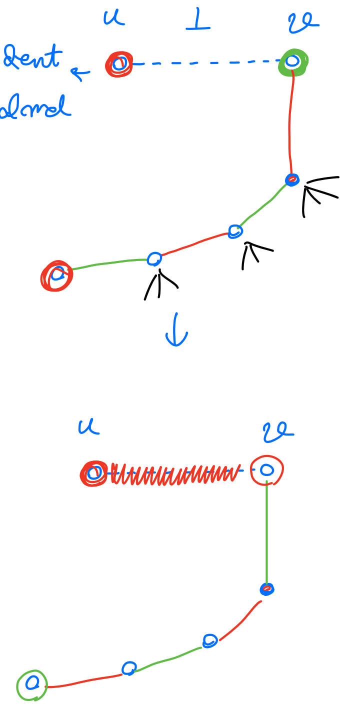
Start with $\chi(u,v) = \perp$ for all $(u,v) \in E$.

while $\exists\, (u,v)$ with $\chi(u,v) = \perp$:

Assign a color from $\{1, \dots, \Delta\}$ to (u,v).



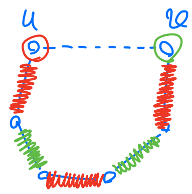
no edge incident to u is colored red



If u,v miss the same color, assign it to (u,v), o/w take the (miss(v), miss(u))-alternating path from v, flip it, and then assign color miss(u) to (u,v).

Fact: A graph is bipartite iff it does not have any odd cycles

This means that the alternating path cannot end in u for bipartite graphs. $\Rightarrow$ Can always extend coloring to an uncolored edge in bipartite graphs. $\square$

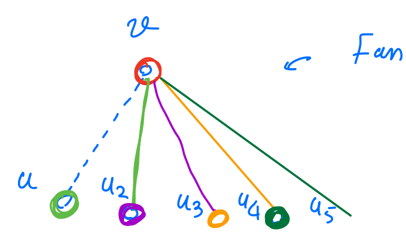


Thm [Vizing '64]: Every graph of max degree Δ admits a $\Delta+1$ edge coloring.

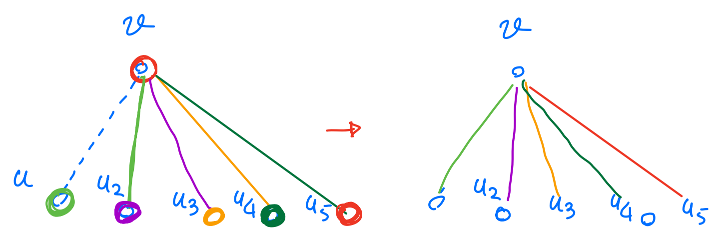
Start with $\chi(u,v) = \perp$ for all $(u,v) \in E$.

while $\exists\, (u,v)$ with $\chi(u,v) = \perp$:

Assign a color from $\{1, \dots, \Delta+1\}$ to (u,v).



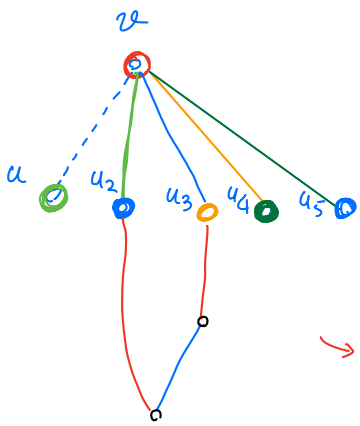
Case 1: We may stop at u_k if v is missing color missed at u_k.

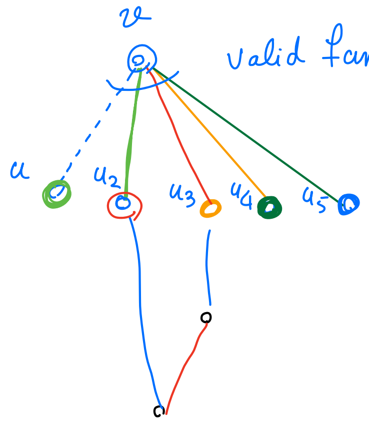


Case 2: missing color of u_k is the same as some u_i for $i < k$.

Case 2-1: The red-blue alternating path ends in u_2.

Flip the a.p.

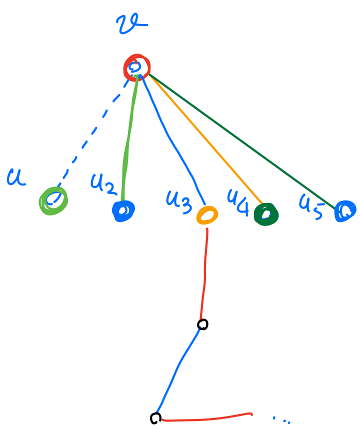


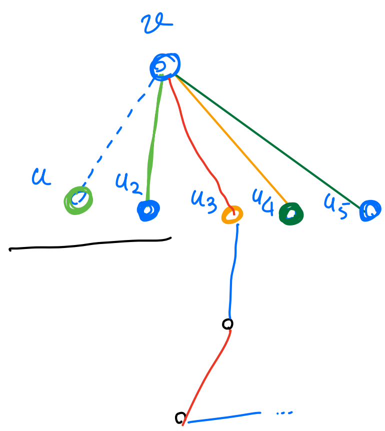


valid fan with the last vertex missing the same color as the center

↑ Case 1

Case 2-2: The red-blue path does not end in u_2.





Now the fan up to u_2 is a case 1 fan