MAX CLIQUE: Given graph G, int $k \geq 1$
det. if G has a clique of size k.

Thm: MAX CLIQUE is NP-complete.

Note that MAX CLIQUE is in NP, since we can verify if subset $U \subseteq V$ is a clique in poly time.

Now we prove MAXCLQ is NP-hard via reduction to IndSet.

Let $\bar{G}$ be the complement of graph G,

i.e. $e \in E(\bar{G})$ iff $e \notin E(G)$.

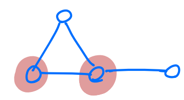
$U \subseteq V$ is an independent set in G iff U is a clique in $\bar{G}$.

So suppose we have an algorithm A for max clique. Then given an instance G for MaxInd, we flip G to obtain $\bar{G}$ in $O(n^2)$ time. Then run A on $\bar{G}$.

Return YES iff $A(\bar{G})$ returns YES. □

MIN VERTEX COVER:

A subset $U \subseteq V$ of vertices is a vertex cover if every edge e has at least one endpoint in U.

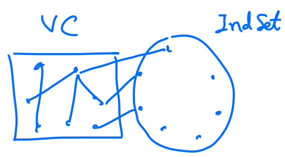


Thm: MIN VC is NP-complete.

Pf: Given $U \subseteq V$ we can verify that U is indeed a VC by going over all edges and verifying that they have an endpoint in U.

So MINVC is in NP

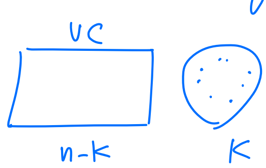
FACT: $U \subseteq V$ is a VC iff $V \setminus U$ is an IndSet.



Given a graph $G=(V,E)$ and $k \geq 1$, to determine if G has an IndSet of size k, we first run MINVC of G with parameter $n-k$.

Return YES iff the answer is YES.

If return YES: There is a VC of size $n-k$, the complement is an IndSet of size $n-(n-k)=k$.



If return NO: Suppose there was an IndSet of size k in G, then the complement would be a VC of size $n-k$. Since the answer is NO, no VC of size $n-k$ exists, therefore no IndSet of size k can exist. □

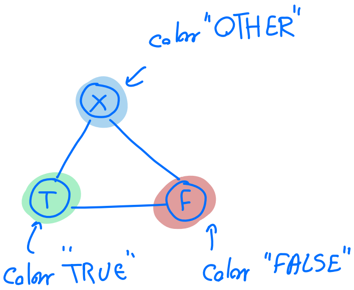
3-COL: Given a graph G, determine if we can color vertices of G using 3 colors s.t. all edges have different colors on their endpoints.

Thm: 3-COL is NP-complete.

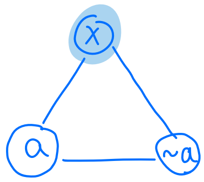
Pf: 3-COL $\in$ NP: Given any assignment of colors we can check if there is any monochromatic edge by simply going over all edges in $O(m)$ time.

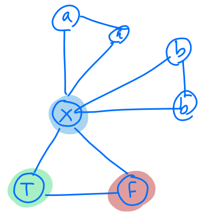
3-COL $\in$ NP-Hard:

First gadget:

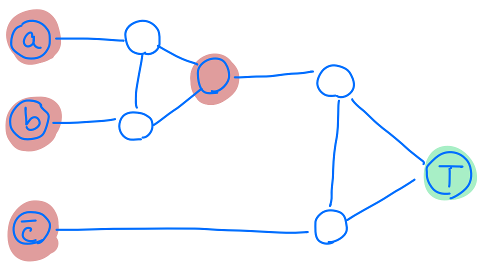


For every variable a in the 3SAT instance we add the gadget





For every clause e.g. $a \vee b \vee \sim c$ we add the following gadget:

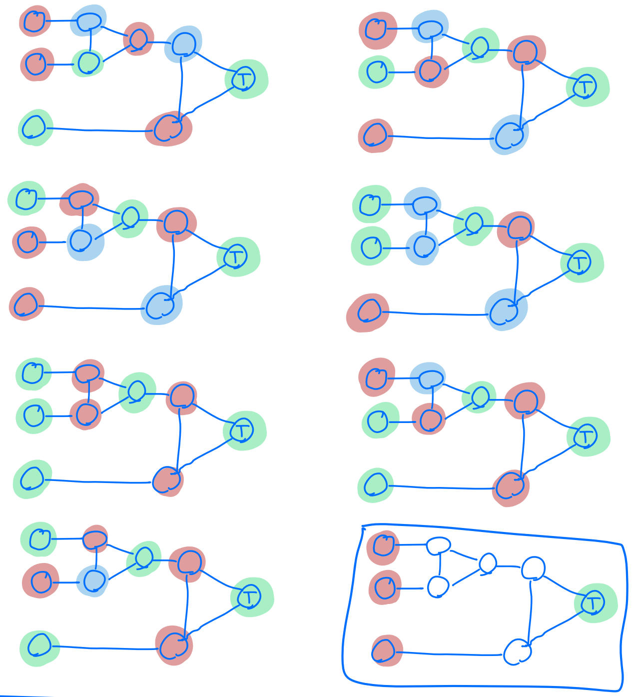


we return YES iff this graph is 3-colorable

3SAT is YES $\Rightarrow$ 3COL is YES

If variable a is TRUE, we assign color "TRUE" to vertex a, "FALSE" to vertex $\bar{a}$

Take gadget



Can't happen

3COL is YES $\Rightarrow$ 3SAT is YES

Simply take the coloring & assign a variable a to be TRUE iff its color is TRUE.

We know that the three input vertices of the clause gadgets cannot be colored "FALSE", so all clauses must be satisfied. ∎