Approximation Algorithms (Part II)

Traveling Salesperson Problem (TSP)

Input: Edge Weighted graph $G=(V,E,w)$

Output: A Hamiltonian cycle of min weight

w.l.o.g. we can assume G is the complete graph.

Thm: TSP is NP-complete.

Pf: A solution can be checked to be a valid H.C. of the claimed weight in polytime.

Given an instance of H.C. problem $H=(V,E_H)$ we construct a weighted graph $G=(V,\binom{V}{2})$ where for a pair (u,v):

$$w_{(u,v)} = \begin{cases} 1 & (u,v)\in E_H \\ 2 & (u,v)\notin E_H \end{cases}$$

Now if H has a H.C., then there is a TSP sol'n of weight n. O/w, TSP sol'n must be $\geq n+1$. $\square$

Thm: For any function $f(n)$ that's computable in poly(n) time, no $f(n)$-apx of TSP can be obtained in poly(n) time unless $P=NP$.

Pf: In the proof above, set the weights of $(u,v)\notin E_H$ to be $f(n)$. $\square$

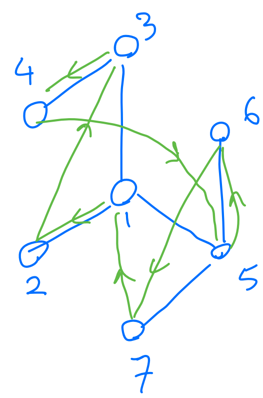
Good News

Metric Assumption: For any 3 disjoint vertices u,v,w, we have

$$w(u,w) \leq w(u,v) + w(v,w).$$

Thm: A 2-apx of metric TSP can be obtained in $O(n^2 \log n)$ time.

Pf: Let M be a minimum spaning tree of the input G which takes $O(n^2 \log n)$ time. We run a DFS on M and return the vertices in the order visited by DFS.



output: 1 2 3 4 5 6 7

A

* Obs: $w(M) \leq OPT$.

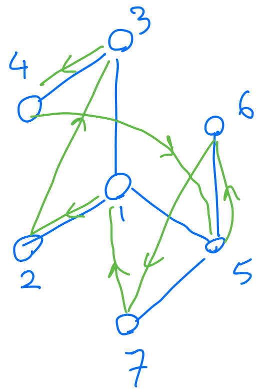
Pf: Take OPT, remove one edge and the result is a spaning tree.

* $w(A) \leq 2 \cdot w(M)$.

Putting the two together implies that

$$w(A) \leq 2w(M) \leq 2\,OPT.$$

Note that a DFS over M goes over each edge of M exactly twice (considering backtrack edges). This implies there is a tour (i.e., a walk that visits every vertex at least once and goes back to the original vertex) of weight $2 \cdot w(M)$.



tour: 1 2 ~~1~~ 3 4 ~~3~~ ~~1~~ 5 6 ~~5~~ 7 ~~5~~ 1

output: 1 2 3 4 5 6 7

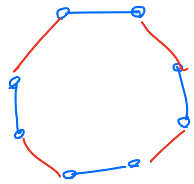
The tour can be turned into a Hamiltonian cycle by shortcutting over already seen vertices. This doesn't increase the cost of the tour because of the metric assumption. □

Suppose that all distances are either 1 or 2. (AKA (1,2)-metric)

Thm: A 1.5-apx of TSP can be obtained in (1,2)-metrics.

pf: Let M be a perfect matching of min total weight. We return an arbitrary H.C. consisting of all edges of M.

Observe that M has weight at most $\frac{TSP}{2}$: Take TSP, it can be decomposed into two perfect matchings by coloring its edges alternatively as "blue" and "red".



One of the two matchings must have weight $\leq \frac{TSP}{2}$, therefore the min cost p.m. also has weight $\leq \frac{TSP}{2}$. Therefore the output is of weight $\leq \frac{TSP}{2} + \frac{n}{2} \cdot 2$.

We also know $TSP \geq n$.

Therefore our sol'n is $\leq 1.5\, TSP$. □

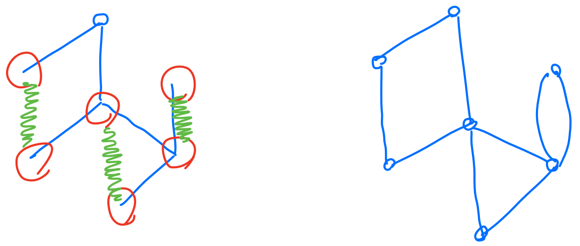
Thm (Christofides '76): A 1.5 apx of TSP can be obtained in polytime even in general metrics.

pf sketch:

Take an MST M of G.

Find a min cost perfect matching P between the odd-degree vertices of M.

Obs: $\underbrace{P \cup M}_{\text{multiset}}$ is a subgraph where every vertex has even degree.



Fact: Any graph with all even-degree vertices has a tour that visits every edge exactly once.

This tour can be turned into H.C. of weight at most $w(P) + w(M)$ using the metric assumption & shortcutting as before.

We already know $w(M) \leq OPT$

we also have $w(P) \leq \frac{OPT}{2}$. □

In 2021 Karlin, Klein, Oveis Gharan showed that a $1.5 - 10^{-36}$ apx of metric TSP can be found in poly time.

Remark: There is fixed $\varepsilon > 0$ s.t. obtaining a $(1+\varepsilon)$-apx of metric TSP is NP-hard.

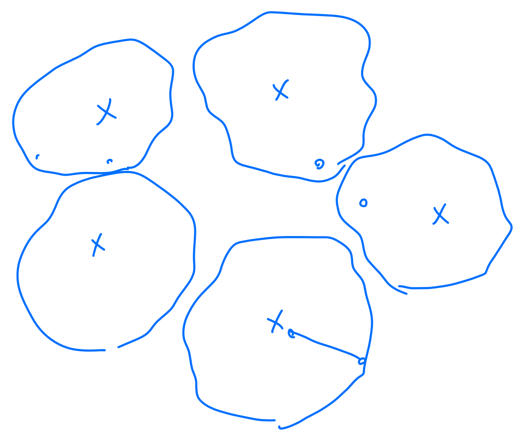
k-center clustering

We are given a complete weighted graph $G = (V, \binom{V}{2}, w)$ and parameter k.

The goal is to select a subset $U \subseteq V$ of size $|U| = k$ such that

$$\max_{v \in V} \min_{u \in U} w(v, u)$$

is minimized.



Remark: The problem is NP-hard to apx within a factor of ~ 1.8 even in metric spaces.

Thm [Gonzalez & Teofilo '85]: metric k-center can be 2-apximated in poly time.

Alg: start with an arbitrary center $u_1 \in V$.

For $i = 2$ to k:

$$u_i \leftarrow \arg\max_{u \in V} \min_{u_i} (u, u_i)$$